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1990 J. Phys. A: Math. Gen. 23 1477

(http://iopscience.iop.org/0305-4470/23/9/012)

J. Phys. A: Math. Gen. 23 (1990) 1477-1492. Printed in the UK

Restricted solid-on-solid models connected with simply laced algebras and conformal field theory

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Received 31 May 1989, in final form 20 July 1989

Abstract. The spectrum of row-to-row transfer matrices in critical RSOS models of A_n type is described explicitly. The thermodynamics of the quantum one-dimensional RSOS model is used to compute values of central charges in corresponding conformal field theories. Generalisation of these results on RSOS related to arbitrary simply laced algebras is discussed.

1. Introduction

In the present paper we study a generalisation of the restricted solid-on-solid (RSOS) model [1]. From the point of view of the underlying algebraic structure, the RSOS model is related to a deformation of the universal enveloping algebra sl(2) [2]. In addition, the RSOS model can be considered as an eight-vertex model with special boundary conditions using a specific construction of intertwining vectors [3, 4]. This construction was recently generalised by Jimbo, Miwa and Okado [5] (JMO) to the case of sl(n) algebra. These authors related the Belavin vertex model [6] with a new interaction-round-a-face (IRF) model whose states are dominant weights of sl(n). In some sense the JMO model corresponds to the simplest vector representation of sl(n). In fact, one can further generalise [7] the JMO model to arbitrary representations using the fusion procedure [7-9]. The Boltzmann weights of these generalised JMO models depend on a rapidity variable u, a temperature-like variable q (elliptic nome), an integer r (similar to those of the RSOS model [1]) and two representations of sl(n) specified by Young diagrams $\lambda = (\lambda_1, \ldots, \lambda_{n-1})$, $\mu = (\mu_1, \ldots, \mu_{n-1})$ such that $\lambda_i \leqslant r-n$, $\mu_i \leqslant r-n$, $i = 1, \ldots, n-1$. The models become critical when $q = 0$. In this case there are at least two physically distinct regimes

$$0 \leqslant u \leqslant \pi/2r \qquad -\pi/2r \leqslant u \leqslant 0. \tag{1.1}$$

We consider the critical case. Using some specific properties of the Boltzmann weights we obtain a system of functional equations which allows us to calculate exactly the spectrum of the transfer matrices $T^{\lambda,\mu}(u)$, generalising the result of [10] where the case $n = 2$ was considered. As usual, the eigenvalues are determinated through the solutions of a system of transcendental equations.

In [11], Belavin, Polyakov and Zamolodchikov have developed a conformal bootstrap program to classify possible types of universal critical behaviour and to calculate

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critical exponents. According to this approach, the critical behaviour of a two-dimensional statistical system at a second-order transition point is described by some unitary [12] conformal field theory, specified by the value of the central charge of the Virasoro algebra of conformal transformations. The parameters of this conformal theory can be extracted from the spectrum of the transfer matrix of the statistical system [13-15].

Consider the one-dimensional quantum model of a chain of N sites defined by a (local) Hamiltonian

$$H = \varepsilon \frac{\mathrm{d}}{\mathrm{d}u} \log T_N^{\lambda,\lambda}(u)|_{u=0} \tag{1.2}$$

where $\lambda = (\underbrace{l, \ldots, l}_{p \text{ times}}, 0, \ldots, 0)$ is the dominant weight of some fixed representation of $\mathrm{sl}(n)$.

In the critical case this Hamiltonian has a gapless spectrum with the linear dispersion law in the vicinity of the Fermi level $\varepsilon(p) = v_F|p - p_F|$. The value of the central charge, c, of the corresponding conformal field theory may be calculated [13-15], on the one hand, from the leading finite-size correction to ground-state energy of the Hamiltonian (1.2) with periodic boundary conditions

$$E_0 = N\varepsilon_0 - \frac{\pi c v_F}{6N} + \mathrm{O}\left(\frac{1}{N^2}\right) \tag{1.3}$$

and, on the other hand, from the low-temperature asymptotics of the specific free energy of the quantum system with the Hamiltonian (1.2) at $N \to \infty$

$$\begin{aligned} &\mathrm{Tr}\, \mathrm{e}^{-\beta H} = \mathrm{e}^{-\beta N F(\beta)} \\ &F(\beta) = F_0 - \frac{\pi c}{6 v_F} \beta^{-2} + \mathrm{O}(\beta^{-2}) \qquad \beta \gg 1 \end{aligned} \tag{1.4}$$

where $\beta = T^{-1}$ is an inverse temperature. We take the second method and investigate the thermodynamics of the model (1.2). In doing this, we use some hypothesis on the types of allowed string solutions to the transcendental equation determining the spectrum of Hamiltonian (1.2) within the thermodynamical limit. The results for the central charges for the two critical regimes $\varepsilon = \pm 1$ corresponding to the different choices of the sign of the Hamiltonian (1.2) are of the form

$$c = \frac{(n^2-1)l}{n+l}\left(1 - \frac{n(n+l)}{r(r-l)}\right) \qquad \varepsilon = -1 \tag{1.5a}$$

$$c = \frac{[(r-n)^2-1]p}{r-n+p}\left(1 - \frac{(r-n)(r-n+p)}{r(r-p)}\right) \qquad \varepsilon = +1. \tag{1.5b}$$

When $n = 2$, formulae (1.5a, b) reproduce the results of [10]. In the case $p = l = 1$, (1.5a) reproduces the result of [16]. The conformal field theories with the latter central charge were constructed in [17]. Note that under the transformation $p \leftrightarrow l$, $n \to r - n$ the expressions (1.5a) and (1.5b) for the above regimes are interchanged. This is a reflection of the duality properties of the Boltzmann weights [16].

We observe that the basic equations of the thermodynamics calculations for the model (1.2) are formulated entirely in algebraic terms (i.e. root system, Cartan matrix,

etc). It is easy to generalise these calculations for the case of D-E-type algebras and calculate the corresponding values of the central charges. Apparently, these results are related to the D-E-type IRF models found in [18]. We have not yet established this relation and claim it as a conjecture.

The paper is organised as follows. In § 2 we formulate the JMO model and discuss the corresponding representation of the Hecke algebra. In § 3 we consider the fusion procedure and calculate the eigenvalues of the transfer matrices. In § 4 the thermodynamics of the one-dimensional model (1.2) is investigated. Section 5 contains the conjectured results for the D-E-type algebras. In this paper we omit most of the details of the calculations. These will be published separately.

2. The model

We consider the JMO model [5]. This is an interaction-round-a-face model on a plane square lattice where the fluctuation variables, assigned to the lattice sites, take values in the set of dominant weights of the sl(n) algebra.

2.1. Basic notation

Let $\{\alpha_i\}$ and $\{\omega_i\}$, $i=1,\ldots,n-1$, be sets of simple roots and fundamental weights of sl(n) algebra [19]:

$$\alpha_i = \varepsilon_i - \varepsilon_{i+1} \tag{2.1}$$

$$\omega_i = \sum_{k=1}^{i} \varepsilon_k - i\varepsilon \qquad \varepsilon = \frac{1}{n}(\varepsilon_1 + \ldots + \varepsilon_n) \tag{2.2}$$

$$\langle \omega_i, \alpha_j \rangle = \delta_{ij} \tag{2.3}$$

where $\langle\,,\rangle$ denotes a scalar product and $\{\varepsilon_i\}$, $i=1,\ldots,n$, is an orthonormal basis in R^n. The weight

$$\hat{\lambda} = \sum_{i=1}^{n-1} m_i\omega_i \qquad m_i \geqslant 0 \tag{2.4}$$

where $\{m_i\}$ are non-negative integers is called dominant. Fix an integer $L \geqslant 1$, and denote by $P_+(n, L)$ the set of dominant weights obeying the requirement

$$\sum_{i=1}^{n-1} m_i \leqslant L. \tag{2.5}$$

We call an element of $P_+(n, L)$ a *local state*. It can be visualised by a Young diagram of depth $\leqslant n$. For a diagram $\lambda = (\lambda_1, \ldots, \lambda_n)$ with $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_n \geqslant 0$ and $\lambda_1 - \lambda_n \leqslant L$, where λ_i is a number of nodes in the ith row, we set

$$\hat{\lambda} = \sum_{i=1}^{n-1} (\lambda_i - \lambda_{i+1})\omega_i = \sum_{i=1}^{n} \lambda_i e_i \in P_+(n, L) \tag{2.6}$$

where

$$e_i = \varepsilon_i - \varepsilon \tag{2.7}$$

with ε defined by (2.2). Two diagrams represent the same element of $P_+(n, L)$ if and only if one is obtained from the other by removing a column of length n.

An ordered pair of local states (a, b), $a, b \in P_+(n, L)$, is called admissible if

$$b = a + e_i \qquad \text{for some } i = 1, \ldots, n. \tag{2.8}$$

For each such pair we place an arrow from a to b, to make $P_+(n, L)$ an oriented graph (figure 1).

Let ρ be a half of the sum of positive roots of sl(n) [19]

$$\rho = \sum_{k=1}^{n-1} \omega_k = \frac{1}{2} \sum_{k=1}^{n} (n-2k+1)\varepsilon_k = \frac{1}{2} \sum_{k=1}^{n-1} k(n-k)\alpha_k \tag{2.9}$$

where α_k, ε_k and ω_k were defined by (2.1)-(2.3). For the state $\hat{\lambda}$ given by (2.6), introduce

$$\lambda_{ij} = |(\hat{\lambda} + \rho, \varepsilon_i - \varepsilon_j)| = \begin{cases} j - i + \lambda_i - \lambda_j & i < j \\ i - j + \lambda_j - \lambda_i & i \geqslant j. \end{cases} \tag{2.10}$$

2.2. Boltzmann weights

Consider a square lattice $\mathscr{L}$ of M rows and N column. With each site $x \in \mathscr{L}$ associate a 'fluctuation variable' which takes values in the set $P_+(n, L)$.

Let v_1, v_2 be unit lattice vectors (directed right and down, respectively). We say that the configuration of states of the whole lattice is admissible if any pair of the adjacent local states of the form $(\lambda_x, \lambda_{x+v_i})$, $x \in \mathscr{L}$; $i = 1, 2$ is admissible in the sense of (2.8).

Using this definition one can show, that for toroidal boundary conditions there are no admissible configurations of the whole lattice, unless

$$M = nM_1 \qquad N = nN_1 \tag{2.11}$$

where M_1, N_1 are integers.

Let $W\left(\begin{smallmatrix} d_f & c_f \\ a_f & b_f \end{smallmatrix}\right)$ denote the Boltzmann weight corresponding to a configuration a_f, b_f, c_f, d_f round a face f (ordered anticlockwise from the SW corner, as shown in figure 2).

The partition function is defined as the sum over all the admissible lattice configurations with the weight equal to the product of Boltzmann weights over all lattice faces

$$Z = \sum_{\{\lambda\}} \prod_{f \in \mathscr{L}} W\begin{pmatrix} d_f & c_f \\ a_f & b_f \end{pmatrix}. \tag{2.12}$$

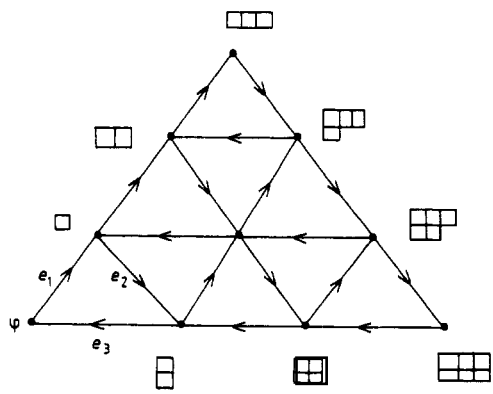

Figure 1. The graph representation of the set of local states $P_+(3, 3)$ $(n = 3, \rho = 3)$. The admissible pairs of the states (a, b) are shown by the arrows from a to b.

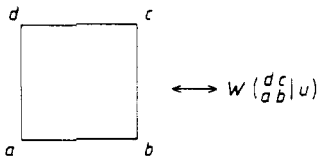

Figure 2. Correspondence between states and vertices.

The non-zero weights are given by [5]

$$W\begin{pmatrix} \hat{\lambda} & \hat{\lambda}+e_i \\ \hat{\lambda}+e_i & \hat{\lambda}+2e_i \end{pmatrix} = \rho h(1-u) \tag{2.13a}$$

$$W\begin{pmatrix} \hat{\lambda} & \hat{\lambda}+e_i \\ \hat{\lambda}+e_i & \hat{\lambda}+e_i+e_j \end{pmatrix} = \rho h(\lambda_{ij}+u)h(1)/h(\lambda_{ij}) \qquad (i \neq j) \tag{2.13b}$$

$$W\begin{pmatrix} \hat{\lambda} & \hat{\lambda}+e_j \\ \hat{\lambda}+e_i & \hat{\lambda}+e_i+e_j \end{pmatrix} = -\rho h(u)h(\lambda_{ij}+1)/h(\lambda_{ij}) \qquad (i \neq j) \tag{2.13c}$$

where ρ is a normalisation factor, λ_{ij} was defined by (2.10) and

$$h(u) = \sin(u\eta) \prod_{k=1}^{\infty} [(1-2q^{2k}\cos(2\eta u)+q^{4k})(1-q^{2k})] \tag{2.14}$$

$$\eta = \pi/r \qquad r = n+L. \tag{2.15}$$

Note that, up to a factor $2q^{1/4}$, $h(u)$ coincides with a standard elliptic theta function $\theta_1(\eta u|q)$ of argument $u\eta$ and nome q.

In the present paper we shall consider the trigonometric case only, setting $q=0$. Then we have

$$h(u) = \sin u\eta = \sin(\pi u/r). \tag{2.14'}$$

Thus, in the trigonometric case, the Boltzmann weights depend on two integers $n \geqslant 2$, $r \geqslant n+1$ and two continuous parameters ρ and u. Regarding n, r, ρ as constants and u as a variable one can write weights (2.13) in the form $W(\begin{smallmatrix} d & c \\ a & b \end{smallmatrix}|u)$. For the particular case $n=2$, the JMO model coincides with the RSOS model of Andrews, Baxter and Forrester [1].

2.3. Hecke algebra

Let $(a_0, \ldots, a_{N+1})$ be a sequence of local states, such that any pair (a_i, a_i+1) is admissible in the sense of (2.8). Define a set of operators $W_i(u)$, $i=1, \ldots, N$, acting in the space of these sequences

$$W_i(u)^{b_0,\ldots,b_{N+1}}_{a_0,\ldots,a_{N+1}} = \left(\prod_{j \neq i} \delta_{a_j,b_j}\right) W\left(\begin{matrix} b_{i-1} & b_i \\ a_i & a_{i+1} \end{matrix} \middle| u\right). \tag{2.16}$$

With these operators the Yang-Baxter equation (YBE) takes the form

$$W_i(u)\,W_{i+1}(u+v)\,W_i(v) = W_{i+1}(v)\,W_i(u+v)\,W_{i+1}(u). \tag{2.17}$$

Using (2.11)-(2.14') one can show that

$$W_i(u) = e^{i\eta u} - e^{-i\eta u} R_i \sin(u\eta)/\sin\eta \tag{2.18}$$

where R is independent of u and

$$R_i R_{i+1} R_i = R_{i+1} R_i R_{i+1}$$
$$R_i^2 - (q-1)R_i - q = 0 \tag{2.19}$$
$$[R_i, R_j] = 0 \qquad |i-j| \geqslant 2.$$

In fact the first two relations are equivalent to the YBE (2.17) for $W_i(u)$ of the form (2.18), and the third is obvious. At this point one recognises that (2.19) gives a representation of the Hecke algebra H_{N+1} [21]. Actually, we can say more. Namely, we obtain the representation of the quotient of the Hecke algebra in which the following relations are satisfied:

$$P^-_{n+1}(i, \ldots, i+n) = 0 \qquad \forall i \tag{2.20}$$

$$P^+_k(i, \ldots, i+k-1) = 0 \qquad \forall i \qquad k = L+1, \ldots, n+L-1 \tag{2.21}$$

where the integers n, L are the parameters of the model defined in section 2.1. Here $P^{\pm}_m$ is the q analogue of the full Young (anti-)symmetriser [7, 9]

$$(P^{\pm}_m)^2 = P^{\pm}_m \tag{2.22}$$

$$P^{\pm}_{m(1,\ldots,m)} = \prod_{k=1}^{m-1} \left(\prod_{l=1}^{m-k} \tilde{W}_l(\pm l) \right) \tag{2.23}$$

$$\tilde{W}_l(\pm l) = \sin \nu \, W_l(\pm l) / \sin(l+1)\nu. \tag{2.24}$$

The ordered product in (2.23) is defined as

$$\prod_{k=1}^{m} A_k = A_1 \ldots A_m. \tag{2.25}$$

One can prove the relations (2.20), (2.21) using the explicit form of the Boltzmann weights (2.13). As is well known [25], the solutions of the Yang-Baxter equations described above are connected with quantised universal enveloping algebras $U_q(\mathrm{sl}(n))$, but here we do not discuss this.

3. The eigenvalues of the transfer matrices

Starting from the model described in the previous section one can construct, using the fusion procedure [7-9], a class of more general models [7]. These models have the fluctuation variables both on sites and edges. Fix two representations of $\mathrm{sl}(n)$ with dominant weights $\Omega_i \in P_+(n, L)$, $i = 1, 2$. Consider the configuration shown in

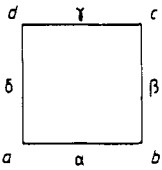

Figure 3. State corresponding to the Boltzmann weight $W^{\Omega_1,\Omega_2}\binom{d\ c}{a\ b}^{\gamma\beta}_{\alpha\delta}$.

figure 3 and denote the corresponding Boltzmann weight as $W^{\Omega_1,\Omega_2}\left(\begin{smallmatrix} d & c \\ a & b \end{smallmatrix}\right)^{\gamma\beta}_{\alpha\delta}$. As before $a, b, c, d \in P_+(n, L)$; however, the admissibility condition now reads

$$[\lambda_{x+v_i}] \subset [\lambda_x] \otimes [\Omega_i] = \sum_{\lambda} [\lambda] \otimes w_\lambda \qquad i = 1, 2 \tag{3.1}$$

where λ_x is the state of the site $x \in \mathscr{L}$, i.e., the representation $[\lambda_{x+v_i}]$ should be contained in the decomposition of the direct product $[\lambda_x] \otimes [\Omega_i]$. The edge variable $\alpha_{x,i}$ corresponding to the edge $(x, x+v_i)$ takes on the values in the space of multiplicities $w_{\lambda_{x+v_i}}$. It fixes the embedding of $[\lambda_{x+v_i}]$ in $[\lambda_x] \otimes [\Omega_i]$. If Ω_i is a symmetric tensor or antisymmetric tensor, the decomposition (3.1) is multiplicity free and all the edge variables for edges directed along v_i are irrelevant. The weights (2.13) correspond to the case $\Omega_1 = \Omega_2 = \omega_1 = (1, 0, \ldots)$.

Introduce some notation. Let $\lambda = (\lambda_1, \ldots, \lambda_{n-1})$ be a Young diagram with $|\lambda| = \Sigma \lambda_i$ nodes. Define a shift tableau B_λ by writing the integer

$$b^\lambda(i, j) = \lambda_1 - i + j - 1 \tag{3.2}$$

at each node (i, j) of the diagram (i and j indicate row and column, respectively). Define a hook length at $x = (i, j) \in \lambda$

$$h^\lambda(i, j) = \lambda_i + \lambda'_j + i - j - 1 \tag{3.3}$$

and a hook length of λ

$$h^\lambda = \lambda_1 + \lambda'_1 - 1 \tag{3.4}$$

where λ'_j is the length of the jth column of λ. Obviously, the maximal element of B_λ is equal to $h^\lambda - 1$. Let $\{u^\lambda\} = (u^\lambda_1, \ldots, u^\lambda_{|\lambda|})$ be a sequence of $b_\lambda(i, j)$ ordered lexicographically in i and j. Fix now two Young diagrams λ and μ and consider the partition function (2.12) for the inhomogeneous lattice of $(|\lambda|+1)(|\mu|+1)$ sites shown in figure 4, assuming the boundary states fixed. The numbers $\{u^\lambda_i\}$ and $\{u^\mu_i\}$ near the broken lines show the shift of the rapidity variables, so that the face located at the

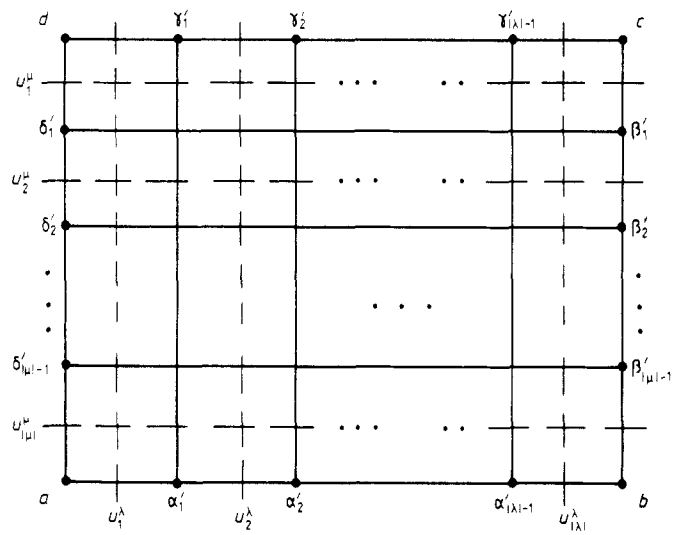

Figure 4. The lattice corresponding to the partition function (3.5).

intersection of the u_i^λth and u_j^μth lines has the weight $W(u-u_i^\lambda+u_j^\mu)$. Denote this partition function as

$$Z\begin{pmatrix} d & c \\ a & b \end{pmatrix}\{\gamma'\}\{\beta'\} = Z\begin{pmatrix} d & c \\ a & b \end{pmatrix}\{\gamma'_1,\ldots,\gamma'_{|\lambda|-1}\}\{\beta'_1,\ldots,\beta_{|\mu|-1}\} \\ \{\alpha'\}\{\delta'\} \qquad\qquad \{\alpha'_1,\ldots,\alpha'_{|\lambda|-1}\}\{\delta'_1,\ldots,\delta'_{|\mu|-1}\} \tag{3.5}$$

Then the weight $W^{\lambda\mu}$ is given by [7]

$$W^{\lambda\mu}\left(\begin{matrix} d & c \\ a & b \end{matrix}\middle| u+\lambda_1-h^\lambda \right)\begin{matrix}\{\gamma\}\{\beta\} \\ \{\sigma\}\{\delta\}\end{matrix}$$
$$= F^{-1}(u-h^\lambda+1)\sum P^{\lambda(a,\{\alpha'\},b)}_{(a,\{\alpha\},b)} P^{\mu(d,\{\delta'\},a)}_{(d,\{\delta\},a)}$$
$$\times Z\begin{pmatrix} d & c \\ a & b \end{pmatrix}\{\gamma'\}\{\beta'\} P^{\lambda(d,\{\gamma\},c)}_{(d,\{\gamma'\},c)} P^{\mu(c,\{\beta\},b)}_{(c,\{\beta'\},b)} \tag{3.6}$$
$$\{\alpha'\}\{\delta'\}$$

where the sum is taken over all the primed states $\{\alpha'\}$, $\{\beta'\}$, $\{\gamma'\}$, $\{\delta'\}$; P^λ (P^μ) is a q analogue of the Young symmetriser corresponding to diagram λ (μ). The full symmetriser $P^{m\omega_1}=P_m^+$ and antisymmetriser $P^{\omega_m}=P_m^-$ are given by (2.23). The general expression can be found in [7, 9]. The function $F(u)$ in (3.6) is defined by

$$F(u)=\prod_{x\in\mu} F^\lambda(u+b^\mu(x)) \tag{3.7}$$

$$F^\lambda(u)=\frac{1}{h(u+\lambda_1-1)}\prod_{x\in\lambda} h(u+b^\lambda(x)) \tag{3.8}$$

where $h(u)$ and $b^\lambda(x)$ are defined by (2.14) and (3.2) respectively. Let $T_N^{\lambda\mu}(u)$ be an N-site row-to-row transfer matrix (for periodic boundary conditions) constructed from the weights $W^{\lambda\mu}(u)$. The fusion procedure implies various relations between the transfer matrices $T^{\lambda\mu}$ with different values of μ. For example,

$$T^{\lambda,\omega_1}(u)T^{\lambda,\omega_1}(u+1)=T^{\lambda,2\omega_1}(u)+T^{\lambda,\omega_2}(u). \tag{3.9}$$

The full set of such types of relations contains, in fact, the more general irreducible transfer matrices $T^{\lambda,\mu}$ where μ is a skew Young diagram [9]. We restrict ourselves to the ordinary diagram μ. Excluding these 'unwanted' transfer matrices one can express $T^{\lambda,\mu}$ through the set of the transfer matrices $\{T^{\lambda,k\omega_1}\}$, $k=1,2,\ldots$, corresponding to the symmetric tensors or through the set of the transfer matrices $\{T^{\lambda,\omega_k}\}$, $k=1,\ldots,n$, corresponding to the antisymmetric tensors

$$T^{\lambda,\mu}(u)=\det\| T^{\lambda_1,(\mu_1+j-i)\omega_1}(u+h^\mu(1,\mu_j)-1)\|_{1\leq i,j\leq n} \tag{3.10}$$

$$T^{\lambda,\mu}(u)=\det\| T^{\lambda_1,\omega_{\mu_i'-i+1}}(u+h^{\mu'}(1,\mu_j')-1)\|_{1\leq i,j\leq \mu_1} \tag{3.11}$$

where by definition $T^{\lambda,k\omega_1}(u)\equiv 0$ for $k<0$, $T^{\lambda,k\omega_1}(u)\equiv 1$ for $k=0$, $T^{\lambda,\omega_0}(u)\equiv 1$, $T^{\lambda,\omega_k}(u)\equiv 0$ for $k>n$ or $k<0$. The hook length $h(i,j)$ is defined by (3.3) and $\mu'=(\mu_1',\mu_2',\ldots)$ is a Young diagram obtained from μ by transposition with respect to the NW-SE diagonal. One can show that for the completely antisymmetric representation $\omega_n=(\underbrace{1,\ldots,1}_{n\text{ times}})$

$$T_N^{\lambda,\omega_n}(u)=(D^\lambda(u))^N E \tag{3.12}$$

$$D^\lambda(u)=\prod_{i=1}^n h(u-\lambda_i+i-1) \tag{3.13}$$

where E is the unity operator. The formulae (3.10) and (3.11) are equivalent in the sense that each of them is a consequence of the other. One can consider these formulae as 'quantum analogues' of the second Weyl formula for the characters of sl(n) (see, e.g., [21], equations (3.4), (3.5) of chapter 1). Note that (3.10)-(3.13) hold also for Belavin's vertex model [6] (or for Cherednik's model [22] in the trigonometric case).

A remarkable feature of the considered model is the existence of additional simple relations for the transfer matrices

$$T^{\lambda,k\omega_1} \equiv 0 \qquad k = r-n+1, \ldots, r-1 \tag{3.14}$$

which are the consequences of the vanishing of the corresponding symmetrisers (2.21).

Choose the normalisation $\rho = \exp(i\eta u)$ in (2.13). From the definition (3.6) it follows, that $T_N^{\lambda,\mu}(u)$ is a polynomial in $e^{2i\eta u}$ of a degree $N|\mu|$. Since, the transfer matrices $T_N^{\lambda,\mu}(u)$ form a commuting family

$$[T_N^{\lambda,\mu}(u), T_N^{\lambda,\nu}(v)] = 0 \tag{3.15}$$

their eigenvalues $\Lambda_N^{\lambda,\mu}(u)$ are also polynomial of the same degree which obey the same set of equations (3.10)-(3.14). These properties are sufficient to find all the eigenvalues $\Lambda^{\lambda\mu}(u)$. Omitting the calculations we describe the results.

We introduce some more notation. Define a set of standard tableaux Tab(λ) associated with diagram λ. The tableau $T \in \mathrm{Tab}(\lambda)$ is formed by integers $\{t(x)\}$, $x \in \lambda$, $1 \leqslant t(x) \leqslant n$, written in squares of the diagram so that

$$t(i,j) \geqslant t(i,j+1) \qquad t(i,j) < t(i+1,j). \tag{3.16}$$

Figure 5 shows all the standard tableaux for the diagram $\lambda = (2, 1)$ for $n = 3$.

The eigenvalues $\Lambda_N^{\lambda\mu}$ are given by

$$\Lambda_N^{\lambda\mu} = \sum_{T \in \mathrm{Tab}(\mu)} \prod_{x \in \mu} X^{t(x)}(u + b^{\mu}(x)) \tag{3.17}$$

where $b^{\mu}(x)$ is defined by (3.2),

$$X^i(u) = \omega_i \frac{Q^{i-1}(u-1)Q^i(u+1)}{Q^{i-1}(u)Q^i(u)} f(u-\lambda_i) \qquad i = 1, \ldots, n \tag{3.18}$$

$$Q^0(u) = Q^n(u) \equiv 1 \qquad f(u) = (h(u))^N \tag{3.19}$$

and where $h(u)$ is defined by (2.14),

$$Q^i(u) = \prod_{k=1}^{q_i} h(u - u_k^{(i)}) \tag{3.20}$$

$$q_i = \frac{n-i}{n} N. \tag{3.21}$$

It remains to define the numbers $\{\omega_i\}$ and $\{u_k^{(i)}\}$. One can show that $T^{\lambda,\mu}(u)$ commute with the operator

$$Y \begin{matrix} \{b_1, \ldots, b_N\} \\ \{a_1, \ldots, a_N\} \end{matrix} = \prod_{k=1}^{N} \delta_{a_i \bar{b}_i} \qquad Y^n = 1 \tag{3.22}$$

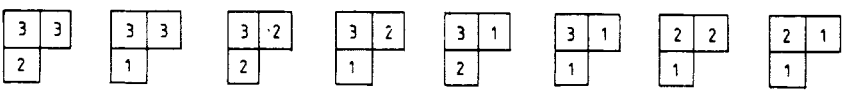

Figure 5. Standard tableaux for the diagram $\lambda = (2, 1)$.

where $\bar{b} = (r-n-b_{n-1}, b_1 - b_{n-1}, \ldots, b_{n-2} - b_{n-1})$. Hence, the eigenvalues $\Lambda^{\lambda,\mu}(u)$ split into sectors labelled by eigenvalue Ω of Y.

The number $\{\omega_i\}$ are arbitrary modulo the relations

$$\omega_i^r = \Omega \qquad \omega_1 \ldots \omega_n = 1 \qquad \omega_i \neq \omega_j \qquad i \neq j \tag{3.23}$$

and the numbers $\{u_i^{(k)}\}$ are the solutions of the system of equations

$$\frac{\omega_i f(u_k^{(i)} - \lambda_i)}{\omega_{i+1} f(u_k^{(i)} - \lambda_{i+1})} = -\frac{Q^{i+1}(u_k^{(i)}+1)Q^i(u_k^{(i)}-1)Q^{i-1}(u_k^{(i)})}{Q^{i+1}(u_k^{(i)}+1)Q^i(u_k^{(i)})Q^{i-1}(u_k^{(i)}-1)} \qquad i = 1, \ldots, n. \tag{3.24}$$

The formula (3.17) for the eigenvalues of the transfer matrix in the corresponding vertex model was obtained by Kirillov and Reshetikhin [26].

4. Thermodynamics of the one-dimensional model

In this section we consider the thermodynamics of the one-dimensional critical quantum model defined by a (local) Hamiltonian

$$H = \frac{\varepsilon}{4\pi} \frac{\mathrm{d}}{\mathrm{d}u} \log T_N^{\lambda,\lambda}(u)|_{u=-p+1} - E_0 \tag{4.1}$$

where $\lambda = (\underbrace{l, \ldots, l}_{p \text{ times}}, 0, \ldots, 0)$, $\varepsilon = \pm 1$, $1 \leq p \leq n-1$, $1 \leq l \leq r-n-1$, and calculate the low-temperature asymptotics of the specific heat capacity. This is equivalent [14, 15] to the calculation of the central charge of an effective conformal field theory describing the critical behaviour of the model.

When $N \to \infty$ and $u \sim (1-p)$ the dominant term in the sum (3.13) corresponds to the tableau T with the integers k in all the squares of the kth row. Thus, we have

$$\begin{aligned} &\Lambda^{\lambda,\lambda}(u)|_{u\sim 1-p} = \Phi(\theta)\frac{Q^p(u+l+p-1)}{Q^p(u+p-1)} + \mathrm{O}(\mathrm{e}^{-\delta N}) \qquad \delta > 0 \\ &\Phi(u) = \prod_{k=0}^{p-1}\prod_{m=0}^{l-1} f(u-l+k+m) \end{aligned} \tag{4.2}$$

where $f(n)$ is given by (3.19). Choosing E_0 in (4.1)

$$E_0 = \frac{\varepsilon}{4\pi}\Phi'(u)|_{u=1-p} \tag{4.3}$$

we obtain for the spectrum of H

$$\begin{aligned} &\mathscr{E} = \varepsilon \sum_{k=1}^{q_p} a_{1,l}(\alpha_k^{(p)}) \\ &u_k^{(p)} = \tfrac{1}{2}(l - \mathrm{i}\alpha_k^{(p)}) \end{aligned} \tag{4.4}$$

$$a_{1,l}(\alpha) = \frac{1}{4\pi\mathrm{i}}\frac{\mathrm{d}}{\mathrm{d}\alpha}\log\frac{\sin(\eta(l+\mathrm{i}\alpha)/2)}{\sin(\eta(l-\mathrm{i}\alpha)/2)} \tag{4.5}$$

where the numbers $\{u_k^{(p)}\}$ are the solutions of the system (3.24).

At $N \to \infty$ these solutions form the strings [23]

$$\begin{aligned} &j\text{-strings: } \alpha_{i,m}^{(k),j} = \beta_i^{(k),j} - \mathrm{i}(j+1-2m) \\ &m = 1, \ldots, j \qquad j = 1, \ldots, r-1 \\ &1^-\text{-strings: } \alpha_i^{(k)(-)} = \beta_i^{(k)(-)} + \mathrm{i}r \end{aligned} \tag{4.6}$$

where $\{\beta\}$ are (real) centres of the strings. We suppose that the number of j-strings, N_j, with $j = r-n+1, \ldots, r-1$, and the number of 1^--strings, N_-, remain finite when $N \to \infty$. More precisely, we suppose that

$$\lim_{N\to\infty} (N_-/N) = \lim_{N\to\infty} (N_j/N) = 0 \qquad j = r-n, \ldots, r-1. \tag{4.7}$$

This conjecture is based on the analogy with the sl(2)-RSOS model [10] and on the analysis of some particular cases (for example, hard hexagons can be considered as the $r=5$, $n=3$ case).

In the thermodynamic limit the centres of the strings form continuous distributions and system (3.24) leads to the following integral equations for densities of the strings $\rho_j^k(\beta)$ and densities of 'holes' $\tilde{\rho}_j^k(\beta)$

$$\delta_{a,p} a_{l,j}^{(r)}(\alpha) = \tilde{\rho}_j^a(\alpha) + \sum_{k=1}^{r-n} \sum_{b=1}^{n-1} A_{jk}^{(r)} * K_{ab}^{(n)} * \rho_k^b(\alpha) \tag{4.8}$$

where $a * b$ denotes the convolution of functions a and b

$$a * b(\alpha) = \int_{-\infty}^{\infty} a(\alpha-\beta) b(\beta)\, \mathrm{d}\beta. \tag{4.9}$$

The functions $a_{j,p}^{(r)}(\alpha)$ and $A_{j,k}^{(r)}(\alpha)$ are defined via their Fourier transforms

$$\hat{a}_{j,p}^{(r)}(x) = \hat{s}(x) \hat{A}_{j,p}^{(r)}(x) \qquad \hat{s}(x) = \frac{1}{2\cosh x} \tag{4.10}$$

$$\hat{A}_{j,k}^{(r)}(x) = 2\hat{a}_j(x) \hat{n}_k(x) \qquad j \geqslant k \tag{4.11}$$

$$\hat{a}_j(x) = \frac{\sinh((r-j)x)}{\sinh(rx)} \qquad \hat{n}_k(x) = \coth x \sinh(kx). \tag{4.12}$$

We use the following normalisation of the Fourier integrals:

$$f(a) = \int_{-\infty}^{\infty} \hat{f}(x)\, \mathrm{e}^{\mathrm{i}ax} \frac{\mathrm{d}x}{2\pi} \qquad f(x) = \int_{-\infty}^{\infty} f(a)\, \mathrm{e}^{\mathrm{i}ax}\, \mathrm{d}a.$$

The function $K_{ab}^{(n)}$ is defined by

$$\hat{K}_{ab}^{(n)}(x) = \delta_{ab} + s(x)(C_{ab} - 2\delta_{ab}) \tag{4.13}$$

where C_{ab} is the Cartan matrix of sl(n). From (3.21) it follows, that

$$\sum_{m=1}^{r-n} m \hat{\rho}_m^a(0) = l(C^{-1})_{ap}. \tag{4.14}$$

This relation together with (4.8) with $j = r-n$ imply that

$$\int_{-\infty}^{\infty} \tilde{\rho}_{r-n}^m(\alpha)\, \mathrm{d}\alpha = 0 \qquad m = 1, \ldots, n-1. \tag{4.15}$$

From the positivity of the density $\tilde{\rho}_{r-n}^m$ this implies $\tilde{\rho}_{r-n}^m \equiv 0$, i.e. for the $(r-n)$-strings the holes are absent in any state.

Excluding now $\rho^m_{r-n}(\alpha)$ from (4.8), inverting the operator $A^{(r-n)}_{kj}$ and using the fact that $\hat{A}^{(r-n)}_{kj}(x) = ((\hat{K}^{(r-n)})^{-1})_{kj}$, we find

$$s\delta_{a,p}\delta_{j,l} = \sum_{b=1}^{n-1} K^{(n)}_{ab} * \rho^b_j + \sum_{k=1}^{r-n-1} K^{(r-n)}_{jk} * \tilde{\rho}^a_k. \tag{4.16}$$

For the energy (4.5) we thus obtain

$$\begin{aligned}\mathscr{E}(\rho) &= \varepsilon\mathscr{E}_0 + \varepsilon \sum_{m=1}^{r-n-1} \int_{-\infty}^{\infty} a^{(r-n)}_{l,m}(\alpha)\rho^p_m(\alpha)\,\mathrm{d}\alpha \\ &= \varepsilon\mathscr{E}'_0 - \varepsilon \sum_{b=1}^{n-1} \int_{-\infty}^{\infty} a^{(n)}_{p,b}(\alpha)\tilde{\rho}^b_l(\alpha)\,\mathrm{d}\alpha \end{aligned} \tag{4.17}$$

where

$$\mathscr{E}_0 = \int_{-\infty}^{\infty} \hat{a}^{(r)}_{r-n,p}(x)\hat{a}^{(n)}_{l,l}(x)\hat{a}^{(r-n)}_{r-n-p}(x)\,\mathrm{d}x$$

$$\mathscr{E}'_0 = \mathscr{E}_0 + \int_{-\infty}^{\infty} a^{(r-n)}_{l,l}(\alpha)a^{(n)}_{p,p}(\alpha)\,\mathrm{d}\alpha.$$

Note the remarkable symmetry of (4.16) and (4.17) under the interchange $n \leftrightarrow (r-n)$, $\varepsilon \to -\varepsilon$, $\rho^a_j \leftrightarrow \tilde{\rho}^a_j$ and $p \leftrightarrow l$. We call this the duality.

The equilibrium state is obtained by a minimisation of the free energy functional

$$F(\rho) = \mathscr{E}(\rho) - TS(\rho)$$

where $S(\rho)$ is the combinatorial entropy of the 'gas' of particles and holes [23].

Omitting the standard calculations we obtain the equilibrium free energy

$$F(T) = -\varepsilon\mathscr{E}_0 - \sum_{j=1}^{r-n-1} \int_{-\infty}^{\infty} a^{(r-n)}_{l,j}(\alpha)\,T\log(1+\mathrm{e}^{-\beta\epsilon^p_j(\alpha)})\,\mathrm{d}\alpha \tag{4.18}$$

where $\beta = 1/T$ and the functions $\epsilon^a_j(\alpha) = T\log(\tilde{\rho}^a_j/\rho^a_j)$ are the solutions of the following system of the nonlinear integral equations:

$$\varepsilon s\delta_{ap}\delta_{jl} = \sum_{m=1}^{r-n-1} K^{(r-n)}_{mj} * T\ln(1+\mathrm{e}^{\beta\epsilon^a_m}) - \sum_{b=1}^{n-1} K^{(n)}_{ab} * T\ln(1+\mathrm{e}^{-\beta\epsilon^b_l}). \tag{4.19}$$

Note again the duality symmetry of these equations.

Using (4.18) and (4.19) one can find the leading term of the low-temperature asymptotics of the entropy. Proceeding exactly as in [10] where the case $n=2$ was considered, we obtain the asymptotics in two regimes:

$$\begin{aligned} S(T) &= -\frac{4\pi T}{3}\left(C^{A_{n-1}}(r-n) - C^{A_{n-1}}(l) - C^{A_{n-1}}(r-l-n) - (n-1)\right) \\ S(T) &= \frac{4\pi T}{3}\left(C^{A_{n-1}}(r-n) - C^{A_{p-1}}(r-n) - C^{A_{n-p-1}}(r-n)\right) \end{aligned} \tag{4.20}$$

where A_{n-1} denotes the root system of $\mathrm{sl}(n)$. For further generalisations to the D and E root systems define $C^{\mathscr{G}}(m)$ for any $A-D-E$ algebra. Let $\{f^a_j\}$, $j = l, \ldots, m-1$, $a = 1, \ldots, \mathrm{rank}(\mathscr{G})$ be solutions to the equations

$$\sum_b C^{\mathscr{G}}_{ab}\log(1-f^b_j) = \sum_p C^{(A_m)}_{jp}\log f^a_p \tag{4.21}$$

where $C^{\mathcal{G}}$ is the Cartan matrix of $\mathcal{G}$. Then

$$C^{\mathcal{G}}(m)=\frac{6}{\pi^2}\sum_{a=1}^{\operatorname{rank}(\mathcal{G})}\sum_{k=1}^{m-1}L(f_k^a) \tag{4.22}$$

where $L(x)$ is the dilogarithmic Rogers function

$$L(x)=-\frac{1}{2}\int_0^x\left(\frac{\log y}{1-y}+\frac{\log(1-y)}{y}\right)\mathrm{d}y. \tag{4.23}$$

As was shown in [24], for $\mathcal{G}=A_{n-1}$,

$$C^{A_{n-1}}(m)=\frac{n(n-1)(m-1)}{m+n}. \tag{4.24}$$

Using the relations

$$C(T)=T\left(\frac{\partial S}{\partial T}\right)=-T\left(\frac{\partial^2 F}{\partial T^2}\right)$$

and (1.4), we obtain

$$C(T)=\frac{\pi c}{3v_{\mathrm{F}}}T+\mathrm{o}(T)=\frac{4\pi c}{3}T+\mathrm{o}(T) \tag{4.25}$$

where $v_{\mathrm{F}}=\frac{1}{4}$ for our normalisations of the Hamiltonian.

Comparing (4.20), (4.24) and (4.25) we readily obtain the result (1.5) for the central charge, c, quoted in § 1. The ground state at $\varepsilon=1$ is the Dirac sea of $(r-n-2)$-strings and at $\varepsilon=-1$ the ground state is the Dirac sea of l-strings. For the energy of the ground state from (4.18) and (4.19) we have:

$$\begin{aligned}&\varepsilon=1 \qquad \mathscr{E}_{\mathrm{GS}}=\mathscr{E}_0\\ &\varepsilon=-1 \qquad \mathscr{E}_{\mathrm{GS}}=-\mathscr{E}_0'\end{aligned} \tag{4.26}$$

where $\mathscr{E}_0$, $\mathscr{E}_0'$ are given after (4.17).

5. Generalisations to the case of simply laced Lie algebras

The calculations of the free energy and central charges were performed in section 4 for the model related to the A_{n-1} algebra. One can formally generalise these calculations to the case of arbitrary A-D-E-type algebras.

It is known [25] that the critical RSOS models, corresponding to the A-type Lie algebras can be obtained from the corresponding vertex models using the q-analogue of Racah-Wigner calculations [26]. This procedure has a generalisation to arbitrary semisimple Lie algebras (in preparation). As we have shown above, the eigenvalues of the transfer matrices of the RSOS model has the same structure as in the corresponding vertex models. The difference is only in the special structure of phases ω_i in (3.18) and in restrictions on the numbers of pseudoparticles q_k (equation (3.21)). Let us suppose that the same picture holds also in the case of D-E-type Lie algebras.

The Bethe equations for the vertex models [27-29] corresponding to simple Lie algebras are known [30]. If the spins in the model are characterised by the highest weight $\lambda=l\omega_p$ of the algebra then the Bethe equations have the form:

$$\left(\frac{\sinh(\pi/2r(a_j^{(s)}+\mathrm{i}l\delta_{sp}))}{(\sinh(\pi/2r(a_j^{(s)}-\mathrm{i}l\delta_{sp}))}\right)^N=\omega_s\prod_{t=1}^{\operatorname{rank}\mathcal{G}}\prod_{k=1}^{q_t}\frac{\sinh(\pi/2r(a_j^{(s)}-a_k^{(t)}+\mathrm{i}C_{st}^{\mathcal{G}}))}{\sinh(\pi/2r(a_j^{(s)}-a_k^{(t)}-\mathrm{i}C_{st}^{\mathcal{G}}))}. \tag{5.1}$$

Our first conjecture is that the eigenvalues of the transfer matrices and the Bethe equations for the RSOS models for arbitrary $\mathcal{G}$ have the same structure as the eigenvalues of the corresponding vertex models (with appropriate phases ω_i) which are known for D_n and E_6 [31]. Second, we suppose that at $N \to \infty$ the solutions of the Bethe equations have the structure (4.6) and (4.7) with n replaced with the dual Coxeter number of $\mathcal{G}$. Define the Hamiltonian of RSOS($\mathcal{G}$) models by the same formula as the sl(n) (4.1). Then, using the result of [30, 31] one can show that the eigenvalues of this Hamiltonian should have the form (4.4), (4.5) with the α determined by (5.1).

After these speculations we can consider the thermodynamics limit in the spectrum of the RSOS($\mathcal{G}$) model. It is not difficult to verify that in the thermodynamic limit, we obtain a system similar to (4.16) for the densities of pseudoparticles ρ_j^a and holes $\tilde{\rho}_j^a$

$$s\delta_{a,p}\delta_{j,l} = \sum_{b=1}^{\operatorname{rank}\mathcal{G}} K_{ab}^{\mathcal{G}} * \rho_j^b + \sum_{k=1}^{r-g-1} K_{jk}^{(r-g)} * \tilde{\rho}_k^a \tag{5.1}$$

where $K_{ab}^{\mathcal{G}}$ is defined by (4.13) with $C_{ab}^{(n)}$ replaced by the Cartan matrix, and g is the dual Coxeter number of the algebra $\mathcal{G}$. Exactly the same modifications are necessary for the thermodynamic equations (4.19). For the free energy we also obtain the same expressions as (4.18) with n replaced by g.

In the limit $T \to 0$ we obtain the following asymptotics of the entropy for two regimes:

$$\begin{aligned} S(T) &= -\frac{4\pi T}{3}\left(C^{\mathcal{G}}(r-g) - C^{\mathcal{G}}(l) - C^{\mathcal{G}}(r-l-g)\text{-rank }\mathcal{G}\right) \qquad \varepsilon = -1 \\ S(T) &= \frac{4\pi}{3}\left(C^{\mathcal{G}}(r-g) - \sum_{\alpha} C^{\mathcal{G}_\alpha}(r-g)\right) \qquad \varepsilon = +1 \end{aligned} \tag{5.2}$$

where g is the dual Coxeter number, $C^{\mathcal{G}}(m)$ are defined by (4.22), the algebras $\mathcal{G}_\alpha$ are subalgebras of $\mathcal{G}$. The Dynkin diagrams of $\mathcal{G}_\alpha$ are obtained from Dynkin diagram of $\mathcal{G}$ after removing the vertex corresponding to the fundamental weight ω_p. An example for $\mathcal{G} = D_n$ is given in figure 6. We verify by numerical calculations the following identity (for D-E-type algebras):

$$C^{\mathcal{G}}(m) = \frac{m \dim \mathcal{G}}{m+g} - \operatorname{rank}\mathcal{G}. \tag{5.3}$$

Remember that for $\mathcal{G} = A_{n-1}$ this identity was analytically proved early [24]. Moreover A N Kirillov has informed us that he has an analytical proof of this identity for $\mathcal{G} = D_n$.

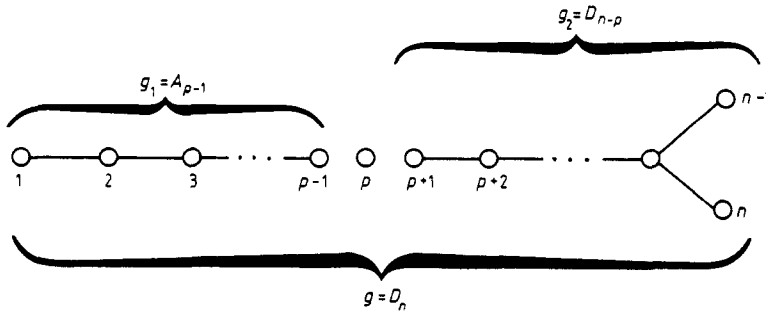

Figure 6.

So, we obtain that the scaling properties of RSOS($\mathcal{G}$) models discussed above are described by conformal field theories with central charges corresponding to the Goddard-Kent-Olive construction:

$$c = -(C^{\mathcal{G}}(r-g) - C^{\mathcal{G}}(l) - C^{\mathcal{G}}(r-l-g) - \text{rank } \mathcal{G}) \qquad \varepsilon = -1$$

$$c = C^{\mathcal{G}}(r-g) - \sum_{\alpha} C^{\mathcal{G}_\alpha}(r-g) \qquad \varepsilon = +1.$$

The ground states for RSOS($\mathcal{G}$) models for simply laced algebras have the same structure as the RSOS(A_{n-1}) model. For $\varepsilon = 1$ this Dirac sea consists of $(r-g)$-strings, for $\varepsilon = -1$ it consists of l-strings. The energy of ground state can be obtained from the thermodynamics of the model and it has the same structure as for $\mathcal{G} = A_{n-1}$.

6. Conclusion

In this paper we considered the RSOS models connected with simply laced algebras. We do not consider an inhomogeneous model. But it is not difficult to see that if we have a chain of spins with representations $(l_1, p_1) \ldots (l_k, p_k)$, we obtain a model corresponding to the coset construction of the type $\mathcal{G}/\mathcal{G}_1 \times \ldots \times \mathcal{G}_m$. So we can hope that the RSOS($\mathcal{G}$) models in the scaling limit give all known conformal field theories. All the details omitted here will be given in the extended version of this paper, where we shall consider also a connection between the RSOS($\mathcal{G}$) models and integrable off-critical specialisations of the conformal field theories.

Acknowledgments

We would like to thank L Faddeev, A Kirillov, F Smirnov and A Zamolodchikov for useful discussions. One of us (NR) also would like to thank Ling Lie Chau and W Nahm for hospitality in the University of California at Davis.

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